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A Complex Frobenius Theorem

LOUIS NIRENBERG

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by

Louis Nirenberg

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A COMPLEX FROBENIUS THEOREM

Louis Nirenberg

1. It was recently shown by Newlander and Nirenberg [3] that a sufficiently differentiable almost complex manifold satisfying the so-called complete integrability conditions is in fact a complex analytic manifold. This result can be formulated as a special "complex Frobenius Theorem"; in fact if the almost complex structure is real analytic the result follows easily from the theorem of Frobenius.

We recall briefly this formulation - an almost complex manifold is a 2n dimensional real manifold on which there is given a real tensor field h^{μ}_{λ} satisfying $h^{\mu}_{\lambda}h^{\lambda}_{\nu} = -\delta^{\mu}_{\nu}$; here summation convention is employed. The problem of reducing this to a complex analytic structure is that of finding new local coordinates $x = (x^1, \dots, x^{2n})$ so that operating with the tensor h^{μ}_{λ} is equivalent to transforming the form $dx^a + idx^{a+n}$ into $i(dx^a + idx^{a+n})$, $a = 1, \dots, n$, i.e. so that in the new coordinate system we have $h^{a}_{a+n} = 1$, $h^{a+n}_{a} = -1$, $a = 1, \dots, n$, with $h^{\mu}_{\lambda} = 0$ otherwise. The complete integrability conditions are easily seen to be necessary: We may suppose that the h^{μ}_{λ} have the special values above at some particular point, in a neighborhood of which we have coordinates $y = (y^1, \dots, y^{2n})$. Then $dx^a + idx^{a+n} = (x^a_{\mu} + ix^{a+n}_{\mu})dy^{\mu}$ and $i(dx^a + idx^{a+n})$ $= (x^a_{\mu} + ix^{a+n}_{\mu})h^{\mu}_{\lambda}dy^{\lambda}$, $a = 1, \dots, n$; here $x^{\lambda}_{\mu} = \partial x^{\lambda}/\partial y^{\mu}$. One then verifies easily that the linear space of forms given by linear combinations of the $dx^a + idx^{a+n}$, $a = 1, \dots, n$, with complex

coefficients is equivalent to the space of forms Ω spanned by the forms $(h^\mu_\lambda + i\delta^\mu_\lambda)dy^\lambda$, $\mu = 1,\ldots,2n$, of which the first n are independent. As a consequence we have the necessary condition: The exterior differential of any form in Ω may be expressed as a sum of exterior products of forms of Ω with first order forms. This may be expressed as

$$(1) \qquad d\Omega \subset \text{ideal generated by } \Omega \quad .$$

We observe also that $\Omega \cap \overline{\Omega} = 0$, where $\overline{\Omega}$ consists of the complex conjugates of the forms in Ω. In [3] these conditions were shown to be sufficient for the equivalence of the almost complex structure to a complex analytic structure - under sufficient differentiability assumptions on the h^μ_λ.

The Frobenius Theorem, we recall (see for instance [2]), asserts that if $\Omega = \Omega(y)$ is a K dimensional subspace of the linear space of real first order differential forms at any point $y = (y^1,\ldots,y^N)$ in a neighborhood in an N dimensional real manifold (varying sufficiently smoothly with y) then a necessary and sufficient condition that we may find new local coordinates x so that $\Omega(y)$ is spanned by $dx^1,\ldots,dx^K$ is that

$$(2) \qquad d\Omega \subset \text{ideal generated by } \Omega \quad .$$

Thus the result above may be considered as a "complex Frobenius Theorem".

Here we prove a general complex Frobenius Theorem containing the above as a special case. The general theorem is, however, easily derived from this special case with the aid of the

real Frobenius Theorem. Before formulating the theorem we state a result equivalent to the real Frobenius theorem above:

Theorem A: *Suppose, there, that for* $j = 1,\ldots,K$, $dy^j =$ *linear combination of* $dy^{K+1},\ldots,dy^N$ (*mod* Ω), *with coefficients that are functions of all the* y^k. *Then* (2) *is necessary and sufficient that the system of differential equations* $\Omega = 0$ *have a unique solution* $y^j(y^{K+1},\ldots,y^N)$, $j = 1,\ldots,K$, *having arbitrary prescribed initial values* $y^j(0,\ldots,0)$ (see for example [2]).

Since our complex Frobenius theorem is local we shall assume that we are operating in a neighborhood of the origin of N space $y = (y^1,\ldots,y^N)$. For convenience we shall also suppose that the coefficients of the forms occurring are of class C^∞. $\overline{\Omega}$ will denote the space of forms which are complex conjugates of the forms of a space Ω.

Theorem 1: *Let* $\Omega = \Omega(y)$ *be a* K *dimensional subspace of the space of complex valued forms defined at every point* y (*and varying in a* C^∞ *way*) *in a neighborhood of the origin. Set* $\Lambda = \Omega \cap \overline{\Omega}$ *and assume that* K' = *dimension of* Λ *is constant and that* Λ *has a basis with* C^∞ *coefficients* (*we note that necessarily* $K' \geq 2K - N$); *set* $L = K-K'$. *Necessary and sufficient for the existence of new local coordinates* x *so that* Ω *is equivalent to the space spanned by*

$$dx^a + idx^{a+L}, \quad dx^\sigma, \quad a = 1.\ldots.L;\ \sigma = N-K'+1,\ldots,N \quad ,$$

are the conditions

(3) $d\Omega \subset$ ideal generated by Ω

(4) $d\Lambda \subset$ ideal generated by Λ .

We observe that if $\tilde{x}$ is another coordinate system with Ω equivalent to the space spanned by $d\tilde{x}^a + id\tilde{x}^{a+L}$, $d\tilde{x}^\sigma$, $a \le L$, $\sigma > N-K'$, then the coordinates $\tilde{x}^\sigma$ are functions of the x^τ alone, $\sigma, \tau > N-K'$, and for $a \le L$ the coordinates $\tilde{x}^a + i\tilde{x}^{a+L}$ are holomorphic functions of the $x^b + ix^{b+L}$, $b = 1, \ldots, L$, and depend in addition only on the variables x^σ, $\sigma > N-K'$.

The necessity of (3), (4) is immediately verified. We shall describe only the proof of sufficiency.

If $\Omega = \Lambda$, so that $L = 0$ and any form in Ω equals a real form in Ω plus i times another, we see that the theorem is simply the real Frobenius theorem. If N is even, $K = N/2$ and $K' = 0$ the theorem is simply the result described earlier and proved in [3].

It is easily seen that Theorem 1 may be restated as a result concerning first order differential operators with complex coefficients: $\sum a^j \frac{\partial}{\partial y^j}$.

Theorem 1': Let S be a (N-K) dimensional linear space of first order differential operators (spanned by (N-K) such operators with C^∞ coefficients) in a neighborhood of the origin in N space. Let $\bar{S}$ denote the space of operators obtained by replacing the coefficients in the operators of S by their complex conjugates, and let $\tilde{S}$ denote the space of operators spanned by those in S and $\bar{S}$. Assume that the dimension of $\tilde{S} = N-K+L$ is constant, and that $\tilde{S}$ can be spanned by operators with C^∞

coefficients. *Then necessary and sufficient for the existence of new local coordinates x so that S is equivalent to the space of operators spanned by*

$$\frac{\partial}{\partial x^{a}} + i\,\frac{\partial}{\partial x^{a+L}}\,, \quad \frac{\partial}{\partial x^{\sigma}}\,, \qquad a = 1,\ldots,L\;; \quad \sigma = L+K+1,\ldots,N$$

are the conditions:

(3') *the commutator of any two operators in S belongs to S,*

(4') *the commutator of any two operators in $\tilde{S}$ belongs to $\tilde{S}$.*

A special case of this, for $N-K = L = 1$, was announced in [3], p. 393.

Before proving Theorem 1 a few remarks about the case N even, $K = N/2$, $K' = 0$, treated in [3], where, the new coordinates x were obtained by solving a system of integral equations in a neighborhood U of the origin: (a) If a given system of basis elements of Ω depends differentiably (say C^{∞}) on some parameters then the new coordinates x so constructed are also continuously differentiable in these parameters (also C^{∞}) provided U is chosen sufficiently small (this was stated at the end of [3]). Indeed it is not difficult to show that the first derivatives of x with respect to the parameters will be as small as desired provided U is sufficiently small.

(b) The new coordinates x constructed in [3] have the property: If, on a neighborhood, a basis for Ω can be chosen so that a finite number of forms $dy^{j} + i dy^{j+K}$ are basis elements, then for these values of j we have $x^{j} = y^{j}$, $x^{j+K} = y^{j+K}$.

In §2 we prove Theorem 1, and in §3 we prove an analogue of Theorem A.

2. Proof of Theorem 1.

2.1. We consider first the case $K' = 0$, i.e. $L = K$, and construct new coordinates x in a neighborhood of the origin so that Ω is the space spanned by

$$dx^a + i dx^{a+L} \quad , \qquad a = 1,\ldots,L \quad .$$

We use summation convention and assume that the indices a,b run from 1 to L, and λ,μ,ν from $2L+1$ to N. By suitably modifying the coordinates y we may suppose that $\Omega(y)$ is spanned by forms $\omega^a(y)$ with $\omega^a(0) = dy^a + i dy^{a+L} \pmod{dy^\lambda}$. In a neighborhood of the origin we may therefore express the dy^a, dy^{a+L} as linear combinations of the forms dy^λ, ω^b, $\overline{\omega^b}$.

To construct the new coordinates we shall (following [3]) consider the y coordinates as functions of the x coordinates, with

$$y^\lambda \equiv x^\lambda \quad .$$

We construct the functions $y^a(x)$, $y^{a+L}(x)$ in two steps: (i) Define them on the $2L$ dimensional submanifold M: $x^\lambda = 0$ for all $\lambda > 2L$. (ii) Extend the functions so defined to a full neighborhood of $x = 0$.

To carry out step (i) consider the forms ω^a on M: $x^\lambda = y^\lambda = 0$; there, near the origin, the forms ω^a, $\overline{\omega^b}$, $a,b = 1,\ldots,L$ are linearly independent. By (3) we have the situation of [3], i.e.

an almost complex structure, and we may therefore choose the y^a, y^{a+L} as functions of x^b, x^{b+L} so that Ω is spanned by the forms $dx^a + idx^{a+L}$ on $x^\lambda = 0$. We remark that the functions as constructed in [3] are such that $\omega^a = dx^a + idx^{a+L}$ at the origin.

To extend the functions y^a, y^{a+L} to a full neighborhood of $x = 0$ we now make the requirement that for every fixed $x^1, \ldots, x^{2L}$, the functions y^a, y^{a+L} satisfy the differential equations $\Omega = 0$ identically in the x^λ. By a remark above this system of equations may be written in the form: dy^a, dy^{a+L} are linear combinations of the $dx^\lambda = dy^\lambda$ with coefficients that are functions of all the y^j. Because of (3) and Theorem A on page 3 there exists for every fixed $(x^1, \ldots, x^{2L})$ a unique solution $y^1, \ldots, y^{2L}$ of this system with given initial values for $x^\lambda = 0$, which we take to be the values determined in (i). We have thus defined a change of coordinates $y(x)$ near $x = 0$, which, in a sufficiently small neighborhood may be seen to have any given number of derivatives and to be nonsingular.

Since the forms of Ω vanish whenever $dx^1 = \ldots = dx^{2L} = 0$, it follows that they are linear combinations of $dx^1, \ldots, dx^{2L}$. It remains still to show that they are linear combinations of the $dx^a + idx^{a+L}$ alone; by (i) this holds on $x^\lambda = 0$. To verify this in general we choose a basis for Ω of the form

$$dx^a + idx^{a+L} + \beta^a_b(dx^b - idx^{b+L}) \quad , \qquad a = 1, \ldots, L$$

with the $\beta^a_b = 0$ on $x^\lambda = 0$. From (3) it follows directly, however, that $\partial\beta^a_b/\partial x^\lambda = 0$, and we conclude that the β^a_b vanish identically. This completes the proof of the theorem for $K' = 0$.

Remarks: (a) The remark (a) at the end of §1 still holds. (b) The remark (b) at the end of that section also holds. Furthermore, we see from our proof that if the coordinates y are such that we have a basis $\omega^a(y)$ of $\Omega(y)$ with $\omega^a(0) = dy^a + idy^{a+L}$ (mod the dy^λ) then we may choose $x^\mu \equiv y^\mu$, $\mu = 2L+1,\ldots,N$.

2.2 We prove the general case of the theorem, $K' \neq 0$, by a reduction to the special case 2.1. It is to be shown that new coordinates x may be found so that Ω is spanned by

(5) $$dx^a + idx^{a+L}\ ,\quad dx^\sigma\ ,\qquad a = 1,\ldots,L;\ \sigma = N-K'+1,\ldots,N\ .$$

We shall assume here that σ runs from $N-K'+1$ to N. Since $\Lambda = \overline{\Lambda}$ we may, in virtue of (4) apply the real Frobenius Theorem, and conclude that we have new coordinates u so that Λ is equivalent to the space spanned by the du^σ. We may write Ω as a direct sum $\Omega = \Lambda \oplus \Gamma$ where Γ is spanned by forms which do not involve the du^σ. Clearly

(6) $$\Gamma \cap \overline{\Gamma} = 0\ ,$$

and, from (3), it follows that if the coordinates u^σ are kept fixed then

(7) $$d\Gamma \subset \text{ideal generated by } \Gamma\ .$$

To prove the general case it suffices to find new coordinates x so that the last K' are unchanged, $x^\sigma \equiv u^\sigma$, and so that when these coordinates are held fixed the space Γ is spanned by

the forms $dx^a + idx^{a+L}$, $a = 1,\dots,L$; for then in general Γ, and hence Ω, is spanned by the forms (5). But if we now regard the x^σ as parameters the problem of making Γ equivalent to the space spanned by $dx^a + idx^{a+L}$ is, because of (6), simply the special case of the theorem treated in 2.1, with N there replaced by $N-K'$, and (3) there being expressed by (7). Thus such a change of coordinates $(u^1,\dots,u^{N-K'}) \to (x^1,\dots,x^{N-K'})$ is possible for every fixed $(u^{N-K'+1},\dots,u^N)$. That this change of coordinates is sufficiently differentiable in the x, and that the full change of coordinates $(u^1,\dots,u^N) \to (x^1,\dots,x^N)$ is nonsingular follows from the remarks (a) at the end of sections 1 and 2.1.

This completes the proof of Theorem 1.

As above, we note that if the given Ω depends differentiably (say C^∞) on some parameters then our new coordinates have continuous derivatives (also C^∞) in these parameters provided we operate in a sufficiently small neighborhood.

In case a given basis for Ω depends analytically on some parameters we may show, for $K' = 0$, the following.

Theorem 2: *Let* $\Omega = \Omega(y)$ *be a space of complex forms as in Theorem* 1, *with* $K' = 0$; *assume that a basis for* Ω *depends analytically on a finite number of real parameters* t *and holomorphically on a finite number of complex parameters* τ, *and that for each fixed* t,τ *(in some* t,τ *neighborhood) condition* (3) *is satisfied. Then we may construct new coordinates* x *satisfying the conditions in Theorem* 1 *so that* $x^a + ix^{a+L}$ *are analytic in* t *and holomorphic in* τ, *for* $a = 1,\dots,L$ *and the coordinates* x^s, $s > 2L$ *are independent of* t *and* τ.

Proof: Because of the analyticity in t we may extend the forms to be holomorphic for complex values of t (still keeping $K' = 0$); thus it suffices to consider the case that a basis for Ω is holomorphic in some complex parameters $\tau = (\tau^1, \ldots, \tau^k)$.

We now extend the system of variables by considering the total system $\tilde{y} = (y^1, \ldots, y^N, \operatorname{Re} \tau^1, \operatorname{Re} \tau^2, \ldots, \operatorname{Im} \tau^k)$, and consider an enlarged system $\tilde{\Omega}$ of forms spanned by Ω and $d\tau^1, \ldots, d\tau^k$. Because of (3), and the holomorphic character of Ω, the enlarged system in $\tilde{y}$ satisfies

$$d\tilde{\Omega} \subset \text{ideal generated by } \tilde{\Omega} \quad .$$

Furthermore $\tilde{\Omega} \cap \overline{\tilde{\Omega}} = 0$. We may therefore apply Theorem 1 and construct new coordinates $\tilde{x} = (x^1, \ldots, x^N, \tilde{x}^1, \ldots, \tilde{x}^{2k})$ with $\tilde{\Omega}$ spanned by

$$dx^a + idx^{a+L} \, , \quad d\tilde{x}^j + id\tilde{x}^{j+k} \, , \quad a = 1, \ldots, L; \; j=1, \ldots, k.$$

By the remark (b) at the end of sections 1 and 2.1 we have

$$\tilde{x}^j + i\tilde{x}^{j+k} = \tau^j \quad .$$

Thus we have $\tilde{\Omega}$ spanned by $dx^a + idx^{a+L}$, $d\tau^j$, $a = 1, \ldots, L$, $j = 1, \ldots, k$. For every fixed τ the new coordinates $x = (x^1, \ldots, x^N)$ are now easily seen to have the properties asserted in the theorem.

An analogous statement should hold for $K' \neq 0$, but this simple argument does not seem to be directly applicable - on going from real to complex values of t the value of K' may change.

3. An analogue of Theorem A. In response to a question of K. Kodaira and D.C. Spencer[1] and as another illustration of the technique used above of extending the number of variables we prove an analogue of Theorem A.

We consider a system of first order partial differential equations for k complex valued functions $(u^1,\ldots,u^k) = u$ of the real variables $(x^1,\ldots,x^n) = x$. For convenience we write $n = 2m+s$ and express the last $2m$ variable $x^{s+1}, x^{s+2}, x^{s+3},\ldots, x^n$ by m complex variables $z^1,\ldots,z^m$, setting $z^\lambda = x^{s+\lambda} + ix^{s+m+\lambda}$, $\lambda = 1,\ldots,m$. In the following the indices j,k will run from 1 to k, the indices p,q from 1 to s, and the indices λ,μ from 1 to m; summation convention will be used.

The system to be studied is

$$(8)\quad \begin{cases} (8)_\lambda & \dfrac{\partial u^j}{\partial \bar{z}^\lambda} + \alpha^j_k \dfrac{\partial \bar{u}^k}{\partial \bar{z}^\lambda} = a^j_\lambda \\[2ex] (8)'_p & \dfrac{\partial u^j}{\partial x^p} + \alpha^j_k \dfrac{\partial \bar{u}^k}{\partial x^p} = b^j_p \end{cases}$$

for $j = 1,\ldots,k$; $\lambda = 1,\ldots,m$; $p = 1,\ldots,s$. Here the coefficients α, a, b are given functions (of class C^∞, for convenience) of the variables u,x, defined in some neighborhood of the origin in the product space. Thus the system is nonlinear. We shall assume that $\alpha(0,0) = 0$ (though it would suffice to assume the $\alpha^j_k(0,0)$ small).

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[1] They recently proved a special case of Theorem A' using methods similar to those used in [3], and asked the author whether it could be deduced from the result in [3]; this we do here.

Theorem A': *The following is necessary and sufficient for the system* (8) *to have solutions* $u(x,u_o)$, *in a neighborhood of* $x = 0$, *with given initial values* $u(0,u_o) = u_o$ *in a neighborhood of the origin in the* u *space, and such that* $u(x,u_o)$ *is of class* C^∞ *in* x *and* u_o: *The space* Ω *of complex Pfaffian forms, in the* 2k+n *variables* (Re u, Im u, x), *which is spanned by the forms*

$$(9) \qquad du^j + \alpha^j_k d\bar{u}^k - a^j_\lambda d\bar{z}^\lambda - b^j_p dx^p \;, \qquad j = 1,\dots,k$$

$$dz^\lambda \;, \qquad \lambda = 1,\dots,m$$

satisfies the conditions

$$(10) \qquad d\Omega \subset \underline{\text{ideal generated by}}\ \Omega \quad .$$

Proof: The necessity of (10) is verified by direct calculation: Differentiate $(8)_\lambda$ with respect to $\bar{z}^\mu$, $(8)_\mu$ with respect to $\bar{z}^\lambda$ and subtract. Form similarly (speaking loosely) $\frac{\partial}{\partial x^p}(8)_\lambda - \frac{\partial}{\partial \bar{z}^\lambda}(8)'_p$ and $\frac{\partial}{\partial x^p}(8)'_q - \frac{\partial}{\partial x^q}(8)'_p$. One sees easily that the resulting relations on the coefficients α, a, b are equivalent to conditions (10).

To prove the sufficiency of (10) we apply Theorem 1, noting that $\Omega \cap \bar{\Omega} = 0$. Applying the theorem, for the case $K' = 0$, to the forms Ω in the 2k+n variables (Re u, Im u, x), we infer that we may introduce 2k+n new variables, which we may write as (Re v, Im v, ξ), so that Ω is spanned by the forms

$$dv^j \;,\quad d\xi^{s+\lambda} + i\,d\xi^{s+m+\lambda} \;, \qquad j = 1,\dots,k; \quad \lambda = 1,\dots,m \quad .$$

Here $v = (v^1,\dots,v^k)$ is complex and $\xi = (\xi^1,\dots,\xi^n)$ is real.

By Remark (b) at the end of §2.1 we may choose $\xi \equiv x$. It follows that the transformation of variables (Re v, Im v) to (Re u, Im u), for fixed x, is nonsingular, and hence that this transformation for $x = 0$ is one-to-one and maps a neighborhood of $v = 0$ onto a neighborhood of $u = 0$, i.e. onto a full neighborhood of initial values u_o for u.

Thus to complete the proof of Theorem A' we need only show that u satisfies (8) as a function of $\xi = x$ and v. To see this write the form (9) in terms of the new variables v, $\xi = x$; this form equals

$$\frac{\partial u^j}{\partial \bar{z}^\lambda} d\bar{z}^\lambda + \frac{\partial u^j}{\partial \bar{v}^\ell} d\bar{v} + \frac{\partial u^j}{\partial x^p} dx^p + a^j_k\left(\frac{\partial \bar{u}^k}{\partial \bar{z}^\lambda} d\bar{z}^\lambda + \frac{\partial \bar{u}^k}{\partial \bar{v}^\ell} d\bar{v} + \frac{\partial \bar{u}^k}{\partial x^p} dx^p\right)$$
$$- a^j_\lambda d\bar{z}^\lambda - b^j_p dx^p$$

modulo the forms of (11). Since this form belongs to Ω the coefficients of $d\bar{z}^\lambda$, $d\bar{v}^\ell$ and dx^p vanish. Setting the coefficients of $d\bar{z}^\lambda$ and dx^p equal to zero we obtain equations (8).Q.E.D.

In conclusion we mention that the complex Frobenius Theorem can be applied to give conditions for a nonanalytic hypersurface in a domain of several complex variables to contain families of (complex) analytic subsurfaces. In studying such questions Behnke and Sommer [1] implicitly assumed a form of the theorem.

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